

The one-loop effective action of noncommutative $\mathcal{N} = 4$ super Yang-Mills is gauge invariant

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Abstract

We study the gauge transformation of the recently computed one-loop four-point function of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $U(N)$. The contributions from nonplanar diagrams are not gauge invariant. We compute their gauge variation and show that it is cancelled by the variation from corresponding terms of the one-loop five-point function. This mechanism is general: it insures the gauge invariance of the noncommutative one-loop effective action.

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Recently noncommutative gauge field theories have attracted much attention [1, 2, 3]. The motivations for such an interest reside primarily on two basic facts: on one side they represent the field theory limit of open strings in the presence of a constant B -field [4], on the other side they offer new examples of non local field theories whose properties can be studied in a rather simple setting [5]. Supersymmetry can be added into the game and in fact the analysis can be performed using a full superspace formulation [6, 7, 8, 9].

In [9] it has been shown that the quantization of noncommutative supersymmetric Yang-Mills theories can be studied within a superfield framework that exactly parallels its commutative counterpart. In particular one can take advantage of the background field method which drastically simplifies perturbative calculations [13, 14]. Indeed in [9, 10] quantum corrections to the noncommutative $\mathcal{N} = 4$ supersymmetric Yang-Mills classical action have been considered. It has been shown that at one loop the noncommutative β -function is zero, since as it happens in the commutative case [15], the two- and three-point contributions to the effective action vanish. Moreover the one-loop four-point function has been computed. The result has been compared with the off-shell continuation of the corresponding result from string amplitudes [11] and complete agreement has been checked [10].

The one-loop four-point function is given by the sum of terms that arise from planar and nonplanar diagrams. The background field formalism introduces the external fields through covariant objects with a multiplication in terms of the $*$ -product. After integration on the loop momentum the contributions from planar graphs are still expressible with the $*$ -product, while the contributions from nonplanar graphs require the introduction of generalized multiplication rules, i.e. $*'$ - and $*_3$ -products [11, 16, 17]. As pointed out in [11, 9, 10] the net result is that while the planar sector respects the gauge invariance of the classical action, the non planar sector is not gauge invariant.

It becomes relevant to investigate whether the Ward identities are satisfied in non-commutative gauge theories. The standard reasoning in commutative gauge theories is that, if the regularized tree-level action is background gauge invariant (or BRS invariant) then the effective action satisfies the background Ward identities (or the Slavnov-Taylor identities). This argument relies on properties like the invariance of the path-integral measure under background gauge (or BRS) transformations which are well established in commutative gauge theories.

In this letter we address the issue of gauge invariance for the effective action of $\mathcal{N} = 4$ super Yang-Mills. We compute the gauge variation of the nonplanar terms in the one-loop four-point function presented in [10] and show that it is cancelled by the variation from corresponding terms of the one-loop five-point function. We discuss the cancellation mechanism in general and argue that it insures the gauge invariance of the noncommutative one-loop effective action. First we set the notation and briefly review the formalism and the results obtained in [9, 10].

The noncommutative $\mathcal{N} = 4$ supersymmetric Yang-Mills classical action written in terms of $\mathcal{N} = 1$ superfields (we use the notations and conventions adopted in [12]) is

given by

$$S = \frac{1}{g^2} \text{Tr} \left(\int d^4x d^4\theta e^{-V} \bar{\Phi}_i e^V \Phi^i + \frac{1}{2} \int d^4x d^2\theta W^2 + \frac{1}{2} \int d^4x d^2\bar{\theta} \bar{W}^2 + \frac{1}{3!} \int d^4x d^2\theta i\epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{1}{3!} \int d^4x d^2\bar{\theta} i\epsilon^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \right) \Big|_* \quad (1)$$

where the Φ^i with $i = 1, 2, 3$ are three chiral superfields, and the $W^\alpha = i\bar{D}^2(e_*^{-V} * D^\alpha e_*^V)$ are the gauge superfield strengths. All the fields are Lie-algebra valued, e.g. $\Phi^i = \Phi_a^i T^a$, in the adjoint representation of $U(N)$. In (1) the symbol $|_*$ indicates that the superfields are multiplied using the $*$ -product defined as

$$(\phi_1 * \phi_2)(x, \theta, \bar{\theta}) \equiv e^{\frac{i}{2}\Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} \phi_1(x, \theta, \bar{\theta}) \phi_2(y, \theta, \bar{\theta}) \Big|_{y=x} \quad (2)$$

The theory can be quantized using the background field method [9], which essentially consists in a non linear splitting between a quantum prepotential V and a background superfield, via covariant derivatives

$$\begin{aligned} \nabla_\alpha &= e_*^{-V} * D_\alpha e_*^V \rightarrow e_*^{-V} * \bar{\nabla}_\alpha^B e_*^V \\ \bar{\nabla}_{\dot{\alpha}} &= \bar{D}_{\dot{\alpha}} \rightarrow \bar{\nabla}_{\dot{\alpha}}^B \end{aligned} \quad (3)$$

On the r.h.s. of (3) the covariant derivatives are expressed in terms of background connections, i.e.

$$\nabla_\alpha^B = D_\alpha - i\Gamma_\alpha \quad \bar{\nabla}_{\dot{\alpha}}^B = \bar{D}_{\dot{\alpha}} - i\bar{\Gamma}_{\dot{\alpha}} \quad \nabla_a^B = \partial_a - i\Gamma_a \quad (4)$$

In this way the external background enters in the quantum action implicitly in the background covariant derivatives through the connections and explicitly in the background field strength $\mathbf{W}_\alpha = \frac{i}{2}[\bar{\nabla}^{B\dot{\alpha}}, \{\bar{\nabla}_{\dot{\alpha}}^B, \nabla_\alpha^B\}]$.

After the splitting the classical action is invariant under two separate sets of gauge transformations.

The quantum transformations:

$$\begin{aligned} e^V &\rightarrow e_*^{i\bar{\Lambda}} * e^V * e_*^{-i\Lambda} & \Phi^i &\rightarrow e_*^{i\Lambda} \Phi^i e_*^{-i\Lambda} \\ \nabla_\alpha &\rightarrow e_*^{i\Lambda} \nabla_\alpha e_*^{-i\Lambda} & \bar{\nabla}_{\dot{\alpha}} &\rightarrow \bar{\nabla}_{\dot{\alpha}} \\ \nabla_\alpha^B &\rightarrow \nabla_\alpha^B & \bar{\nabla}_{\dot{\alpha}}^B &\rightarrow \bar{\nabla}_{\dot{\alpha}}^B \end{aligned} \quad (5)$$

with supergauge parameter Λ which is a covariantly chiral superfield, $\bar{\nabla}_{\dot{\alpha}}^B \Lambda = 0$.

The background transformations:

$$\begin{aligned} e^V &\rightarrow e_*^{iK} * e^V * e_*^{-iK} & \Phi^i &\rightarrow e_*^{iK} \Phi^i e_*^{-iK} \\ \nabla_\alpha &\rightarrow e_*^{iK} \nabla_\alpha e_*^{-iK} & \bar{\nabla}_{\dot{\alpha}} &\rightarrow e_*^{iK} \bar{\nabla}_{\dot{\alpha}} e_*^{-iK} \\ \nabla_\alpha^B &\rightarrow e_*^{iK} \nabla_\alpha^B e_*^{-iK} & \bar{\nabla}_{\dot{\alpha}}^B &\rightarrow e_*^{iK} \bar{\nabla}_{\dot{\alpha}}^B e_*^{-iK} \end{aligned} \quad (6)$$

with a real superfield parameter, $K = \bar{K}$. The quantum gauge invariance needs gauge-fixing and this can be done in a background covariant way adding to the classical Lagrangian background covariantly chiral gauge-fixing functions, $\nabla_B^2 V$ and $\bar{\nabla}_B^2 V$. As emphasized in [9, 10] the quantization for the noncommutative theory proceeds following the same steps as for the commutative case. At one loop, amplitudes with external vector fields receive contributions from quantum V fields only, since the chiral matter loops are exactly cancelled by the ghost loops because of statistics.

The quantum quadratic action, relevant for the computation of the one-loop vector corrections is

$$\begin{aligned}
-\frac{1}{2g^2} \text{Tr} \Big(& V * \frac{1}{2} \partial^a \partial_a V - iV * [\Gamma^a, \partial_a V]_* - \frac{1}{2} V * [\Gamma^a, [\Gamma_a, V]_*]_* \\
& - iV * \{\mathbf{W}^\alpha, D_\alpha V\}_* - V * \{\mathbf{W}^\alpha, [\Gamma_\alpha, V]_*\}_* \\
& - iV * \{\bar{\mathbf{W}}^{\dot{\alpha}}, \bar{D}_{\dot{\alpha}} V\}_* - V * \{\bar{\mathbf{W}}^{\dot{\alpha}}, [\bar{\Gamma}_{\dot{\alpha}}, V]_*\}_* \Big)
\end{aligned} \tag{7}$$

The background connections have been defined in (4) and we have introduced the notation $[A, B]_* = A * B - B * A$. The interactions with the background fields are at most linear in the D spinor derivatives. Therefore one can immediately conclude that the first non vanishing correction to the effective action is given by the four-point function since D -algebra supergraph rules require at least two D 's and two \bar{D} 's in the loop. The complete result has been obtained in [10] and we make reference to that work for details of the calculation. Here we briefly explain how the various contributions are assembled and we present the final answer.

The vertices that enter in the evaluation of the four-point function are

$$\frac{i}{2g^2} \text{Tr} \left(V * \{\mathbf{W}^\alpha, D_\alpha V\}_* + V * \{\bar{\mathbf{W}}^{\dot{\alpha}}, \bar{D}_{\dot{\alpha}} V\}_* \right) \tag{8}$$

When inserted in the loop, depending on the order in which the quantum V 's are Wick contracted, they produce two types of terms, an untwisted P -term and a twisted T -term [9]

$$\begin{aligned}
P &\rightarrow \text{Tr}(T^a T^b T^c) e^{-\frac{i}{2}(k_1 \times k_2 + k_2 \times k_3 + k_1 \times k_3)} = \text{Tr}(T^a T^b T^c) e^{-\frac{i}{2}k_2 \times k_3} \\
T &\rightarrow -\text{Tr}(T^c T^b T^a) e^{\frac{i}{2}(k_1 \times k_2 + k_2 \times k_3 + k_1 \times k_3)} = -\text{Tr}(T^c T^b T^a) e^{\frac{i}{2}k_2 \times k_3}
\end{aligned} \tag{9}$$

where we have defined $k_i \times k_j \equiv (k_i)_\mu \Theta^{\mu\nu} (k_j)_\nu$ and used $k_1 + k_2 + k_3 = 0$.

The external background fields are arranged as follows

$$\begin{aligned}
\mathcal{T}(1a, 2b, 3c, 4d) &= \mathbf{W}^{\alpha a}(p_1) \mathbf{W}_\alpha^b(p_2) \bar{\mathbf{W}}^{\dot{a} c}(p_3) \bar{\mathbf{W}}_{\dot{\alpha}}^d(p_4) \\
&+ \mathbf{W}^{\alpha a}(p_1) \bar{\mathbf{W}}^{\dot{a} b}(p_2) \bar{\mathbf{W}}_{\dot{\alpha}}^c(p_3) \mathbf{W}_\alpha^d(p_4) \\
&- \mathbf{W}^{\alpha a}(p_1) \bar{\mathbf{W}}^{\dot{a} b}(p_2) \mathbf{W}_\alpha^c(p_3) \bar{\mathbf{W}}_{\dot{\alpha}}^d(p_4) + \text{h.c.}
\end{aligned} \tag{10}$$

The above expression, completely symmetric in the exchanges of any couple (1a) \leftrightarrow (2b) \leftrightarrow (3c) \leftrightarrow (4d), can be compared easily with the bosonic result from string amplitude calculations since we have

$$\begin{aligned} & \int d^2\theta \, d^2\bar{\theta} \left[\mathbf{W}^{\alpha a}(p_1) \mathbf{W}_\alpha^b(p_2) \bar{\mathbf{W}}^{\dot{a}c}(p_3) \bar{\mathbf{W}}_{\dot{a}}^d(p_4) \right. \\ & \quad + \mathbf{W}^{\alpha a}(p_1) \bar{\mathbf{W}}^{\dot{a}b}(p_2) \bar{\mathbf{W}}_{\dot{a}}^c(p_3) \mathbf{W}_\alpha^d(p_4) \\ & \quad \left. - \mathbf{W}^{\alpha a}(p_1) \bar{\mathbf{W}}^{\dot{a}b}(p_2) \mathbf{W}_\alpha^c(p_3) \bar{\mathbf{W}}_{\dot{a}}^d(p_4) + \text{h.c.} \right] \\ & \rightarrow t^{\alpha\beta\gamma\delta\mu\nu\rho\sigma} F_{\alpha\beta}^a(p_1) F_{\gamma\delta}^b(p_2) F_{\mu\nu}^c(p_3) F_{\rho\sigma}^d(p_4) \end{aligned} \quad (11)$$

where $t^{\alpha\beta\gamma\delta\mu\nu\rho\sigma}$ is the symmetric tensor given e.g. in (9.A.18) of [18]. The one-loop four-point diagrams contain four propagators which produce a factor

$$I_0(k; p_1, \dots, p_4) = \frac{1}{(k + p_1)^2 k^2 (k - p_4)^2 (k + p_1 + p_2)^2} \quad (12)$$

They can be divided into planar and non planar contributions. There are two planar graphs: one with four P vertices and one with four T vertices with a trace factor $\text{Tr}(T^p T^q T^r T^s)$ from the $U(N)$ matrices. Their contribution is symbolically written as

$$\mathcal{P}(p_1, \dots, p_4) = P_1 P_2 P_3 P_4 + T_1 T_2 T_3 T_4 \quad (13)$$

The nonplanar diagrams contain either two twisted and two untwisted vertices or else one twisted and three untwisted vertices (equivalently three twisted and one untwisted). For the first group the trace on the $U(N)$ matrices gives a factor like $\text{Tr}(T^p T^q) \text{Tr}(T^r T^s)$, while for the second group it gives $\text{Tr}(T^p) \text{Tr}(T^q T^r T^s)$. Their contribution is symbolically written as $\mathcal{A}(k, p_1, \dots, p_4) + \mathcal{B}(k, p_1, \dots, p_4)$ with

$$\begin{aligned} \mathcal{A}(k, p_1, \dots, p_4) = & P_1 P_2 T_3 T_4 + P_1 T_2 T_3 P_4 + P_1 T_2 P_3 T_4 \\ & + T_1 T_2 P_3 P_4 + T_1 P_2 P_3 T_4 + T_1 P_2 T_3 P_4 \end{aligned} \quad (14)$$

and

$$\begin{aligned} \mathcal{B}(k, p_1, \dots, p_4) = & P_1 P_2 P_3 T_4 + P_1 T_2 P_3 P_4 + P_1 P_2 T_3 P_4 + T_1 P_2 P_3 P_4 \\ & + T_1 T_2 T_3 P_4 + T_1 P_2 T_3 T_4 + T_1 T_2 P_3 T_4 + P_1 T_2 T_3 T_4 \end{aligned} \quad (15)$$

Using the above definitions the complete one-loop vector four-point function is given by [10]

$$\begin{aligned} \Gamma_{\text{total}} = & \frac{1}{4} \int d^2\theta \, d^2\bar{\theta} \frac{d^4 p_1 \, d^4 p_2 \, d^4 p_3 \, d^4 p_4}{(2\pi)^{16}} \delta(\sum p_i) \mathcal{T}(1a, 2b, 3c, 4d) \\ & \int d^4 k \, I_0(k; p_1, \dots, p_4) [\mathcal{P}(p_1, \dots, p_4) + \mathcal{A}(k, p_1, \dots, p_4) \\ & \quad + \mathcal{B}(k, p_1, \dots, p_4)] \end{aligned} \quad (16)$$

The above expression can be evaluated in the low-energy approximation $p_i \cdot p_j$ small, with $p_i \times p_j$ finite. In this case the integration on the k -loop momentum can be performed

exactly. Introducing an IR mass regulator in a background gauge invariant manner, and the definition $p \circ p = p_\mu \Theta^{\mu\nu} \Theta_{\rho\nu} p^\rho$, the complete result for the low-energy planar and nonplanar contributions to the four-point function can be written as

$$\begin{aligned}
\Gamma_{\text{l.e.}} = & \frac{\pi^2}{4} \int d^2\theta \, d^2\bar{\theta} \, \frac{d^4p_1 \, d^4p_2 \, d^4p_3 \, d^4p_4}{(2\pi)^{16}} \delta(\sum p_i) \\
& \left[\mathbf{W}^{\alpha\alpha}(p_1) \mathbf{W}_\alpha^b(p_2) \bar{\mathbf{W}}^{\dot{\alpha}c}(p_3) \bar{\mathbf{W}}_{\dot{\alpha}}^d(p_4) + \mathbf{W}^{\alpha\alpha}(p_1) \bar{\mathbf{W}}^{\dot{\alpha}b}(p_2) \bar{\mathbf{W}}_{\dot{\alpha}}^c(p_3) \mathbf{W}_\alpha^d(p_4) \right. \\
& \quad \left. - \mathbf{W}^{\alpha a}(p_1) \bar{\mathbf{W}}^{\dot{\alpha}b}(p_2) \mathbf{W}_\alpha^c(p_3) \bar{\mathbf{W}}_{\dot{\alpha}}^d(p_4) + \text{h.c.} \right] \\
& \left\{ \frac{1}{6m^4} \text{Tr}(T^a T^b T^c T^d) e^{\frac{i}{2}(p_2 \times p_1 + p_1 \times p_4 + p_2 \times p_4)} \right. \\
& \quad + \frac{1}{m^2} \frac{(p_1 + p_2) \circ (p_1 + p_2)}{2} \frac{\sin\left(\frac{p_1 \times p_2}{2}\right)}{\frac{p_1 \times p_2}{2}} \frac{\sin\left(\frac{p_3 \times p_4}{2}\right)}{\frac{p_3 \times p_4}{2}} \\
& \quad K_2(m\sqrt{(p_1 + p_2) \circ (p_1 + p_2)}) \text{Tr}(T^a T^b) \text{Tr}(T^c T^d) \\
& \quad - \frac{1}{m^2} (p_1 + p_2 + p_3) \circ (p_1 + p_2 + p_3) \frac{e^{-\frac{i}{2}(p_1 \times p_2 + p_2 \times p_3 + p_1 \times p_3)}}{(p_1 \times p_4)(p_3 \times p_4)} \\
& \quad K_2(m\sqrt{(p_1 + p_2 + p_3) \circ (p_1 + p_2 + p_3)}) \text{Tr}(T^a T^b T^c) \text{Tr}(T^d) \\
& \quad \left. + \text{h.c.} \right\} \tag{17}
\end{aligned}$$

Again using (11) we can extract the purely bosonic contributions in terms of the $F_{\mu\nu}$'s. The resulting expression coincides with the off-shell extrapolation of the field theory limit obtained in [11] from open string amplitudes. The terms from planar diagrams can be rewritten in terms of \ast -products between the superfield strengths, while the terms proportional to $\text{Tr}(T^a T^b) \text{Tr}(T^c T^d)$ and to $\text{Tr}(T^a T^b T^c) \text{Tr}(T^d)$ corresponding to non planar diagrams reproduce the \ast' - and the \ast_3 -products respectively [16, 11, 17]. While the planar contributions are gauge invariant under the background transformations in (6), the non planar terms break gauge invariance explicitly, despite the fact that they are expressed in terms of covariant objects. We observe that even if we were to replace the \ast' and the \ast_3 operations with \ast -products still the nonplanar terms would not maintain gauge invariance. The new generalized multiplication rules are not the cause of trouble. In fact we will show that gauge invariance is recovered taking into account the gauge variation of corresponding terms from the one-loop five point function.

Now we discuss in some detail the gauge variation of the one-loop four-point function in (16). The total variation consists in the sum of four terms corresponding to the variation of each external background field contained in the expression $\mathcal{T}(1a, 2b, 3c, 4d)$ explicitly given in (10). Symmetry considerations allow to repeat the same argument for each of the

four contributions, so that in order to be specific we consider the variation of the $\mathbf{W}^\alpha(p_1)$ superfield. From (6) we obtain the background gauge transformation of the superfield strength

$$\delta \mathbf{W}^\alpha = iK * \mathbf{W}^\alpha - i\mathbf{W}^\alpha * K \equiv \delta_1 \mathbf{W}^\alpha + \delta_2 \mathbf{W}^\alpha \quad (18)$$

The two terms can be written in momentum space as

$$\begin{aligned} \delta_1 \mathbf{W}^\alpha(p_1) &= i \int d^4x e^{-ip_1 \cdot x} K * \mathbf{W}^\alpha \\ &= i \int d^4x \int \frac{d^4p_0 d^4k}{(2\pi)^8} e^{i(p_0+k) \cdot x} e^{-\frac{i}{2}p_0 \times k} e^{-ip_1 \cdot x} K(p_0) \mathbf{W}^\alpha(k) \\ &= i \int \frac{d^4p_0}{(2\pi)^4} e^{-\frac{i}{2}p_0 \times p_1} K(p_0) \mathbf{W}^\alpha(p_1 - p_0) \end{aligned} \quad (19)$$

and

$$\delta_2 \mathbf{W}^\alpha(p_1) = -i \int \frac{d^4p_0}{(2\pi)^4} e^{\frac{i}{2}p_0 \times p_1} \mathbf{W}^\alpha(p_1 - p_0) K(p_0) \quad (20)$$

Our goal is to prove that such a variation of the external background field \mathbf{W}^α in the four-point function is compensated by the variation of the background connection $\mathbf{\Gamma}_a$ which appears as an additional vertex from (7) in the corresponding five-point function. The analysis is quite simple if it is performed directly on the various one-loop diagrams that contribute to the total answer. As an example of the cancellation mechanism we focus on the contribution in (16) which has vertices in the order $P_1 P_2 T_3 T_4$ with $\mathbf{W}^\alpha(p_1) \mathbf{W}_\alpha(p_2) \bar{\mathbf{W}}^{\dot{\alpha}}(p_3) \bar{\mathbf{W}}_{\dot{\alpha}}(p_4)$ as external background. The corresponding Feynman diagram is shown in Fig. 1.

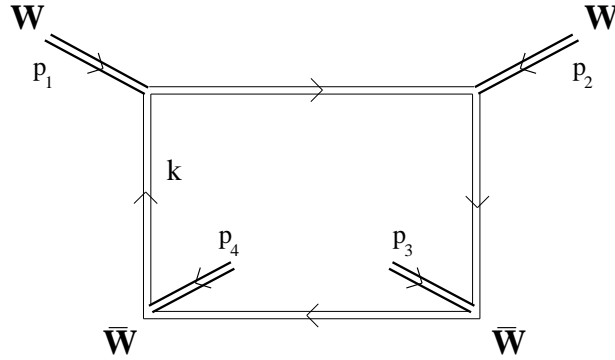


Figure 1: $P_1 P_2 T_3 T_4$ contribution to the four point function

It gives

$$\Gamma_{PT} = \frac{1}{4} \int d^2\theta d^2\bar{\theta} \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16}} \delta(\sum p_i) e^{-\frac{i}{2}(p_1 \times p_2 - p_3 \times p_4)} \quad (21)$$

$$\text{Tr}[\mathbf{W}^\alpha(p_1)\mathbf{W}_\alpha(p_2)] \text{Tr}[\bar{\mathbf{W}}^{\dot{\alpha}}(p_3)\bar{\mathbf{W}}_{\dot{\alpha}}(p_4)] \int d^4k \frac{e^{-ik \times (p_1+p_2)}}{(k+p_1)^2 k^2 (k-p_4)^2 (k+p_1+p_2)^2}$$

Now we compute its δ_1 gauge transformation as in (19)

$$\begin{aligned} \delta_1 \Gamma_{PT}^{(1)} &= \frac{i}{4} \int d^2\theta \, d^2\bar{\theta} \frac{d^4p_1 \, d^4p_2 \, d^4p_3 \, d^4p_4}{(2\pi)^{16}} \delta(\sum p_i) \\ &\quad \int d^4k \frac{e^{-\frac{i}{2}(p_1 \times p_2 - p_3 \times p_4)} e^{-ik \times (p_1+p_2)}}{(k+p_1)^2 k^2 (k-p_4)^2 (k+p_1+p_2)^2} \\ &\quad \int \frac{d^4p_0}{(2\pi)^4} e^{-\frac{i}{2}p_0 \times p_1} \text{Tr}[K(p_0)\mathbf{W}^\alpha(p_1-p_0)\mathbf{W}_\alpha(p_2)] \text{Tr}[\bar{\mathbf{W}}^{\dot{\alpha}}(p_3)\bar{\mathbf{W}}_{\dot{\alpha}}(p_4)] \quad (22) \end{aligned}$$

The $\delta_2 \Gamma_{PT}^{(1)}$ variation is obtained in similar manner. In the same way one compute $\delta \Gamma_{PT}^{(2)}$, $\delta \Gamma_{PT}^{(3)}$ and $\delta \Gamma_{PT}^{(4)}$ varying in (21) $\mathbf{W}_\alpha(p_2)$, $\bar{\mathbf{W}}^{\dot{\alpha}}(p_3)$ and $\bar{\mathbf{W}}_{\dot{\alpha}}(p_4)$ respectively. One finds that the sum $\delta \Gamma_{PT}^{(1)} + \delta \Gamma_{PT}^{(2)} + \delta \Gamma_{PT}^{(3)} + \delta \Gamma_{PT}^{(4)}$ is not zero. Thus we look for a cancellation from a higher-point function.

Let us consider the contribution to the one-loop five point-function corresponding to a supergraph with the same vertices considered above plus the following additional vertex (cf. the action in (7))

$$\mathcal{V} = \frac{i}{2g^2} \int d^2\theta d^2\bar{\theta} d^4x \, \text{Tr} V * [\mathbf{\Gamma}^a, \partial_a V]_* \quad (23)$$

When inserted in the loop, depending on the order in which the quantum fields are Wick contracted it will produce contributions of untwisted or twisted type. Now we compute its gauge variation under the background transformations in (6) and show that its presence compensates the terms in (22). From (6) we obtain the linearized background gauge variation of the connection

$$\delta \mathbf{\Gamma}_a = \partial_a K \quad (24)$$

so that the vertex varies into

$$\begin{aligned} \delta \mathcal{V} &= \frac{i}{g^2} \text{Tr} \int d^2\theta d^2\bar{\theta} d^4x \, (-V * K * \square V + \square V * K * V) \\ &\equiv \delta_1 \mathcal{V} + \delta_2 \mathcal{V} \quad (25) \end{aligned}$$

As emphasized above each term in (25) produces an untwisted and a twisted contribution

$$\delta_1 \mathcal{V} \rightarrow \delta_1 \mathcal{V}_P + \delta_1 \mathcal{V}_T \quad \delta_2 \mathcal{V} \rightarrow \delta_2 \mathcal{V}_P + \delta_2 \mathcal{V}_T \quad (26)$$

We focus on \mathcal{V}_P . The analysis of the other terms leads to corresponding similar conclusions.

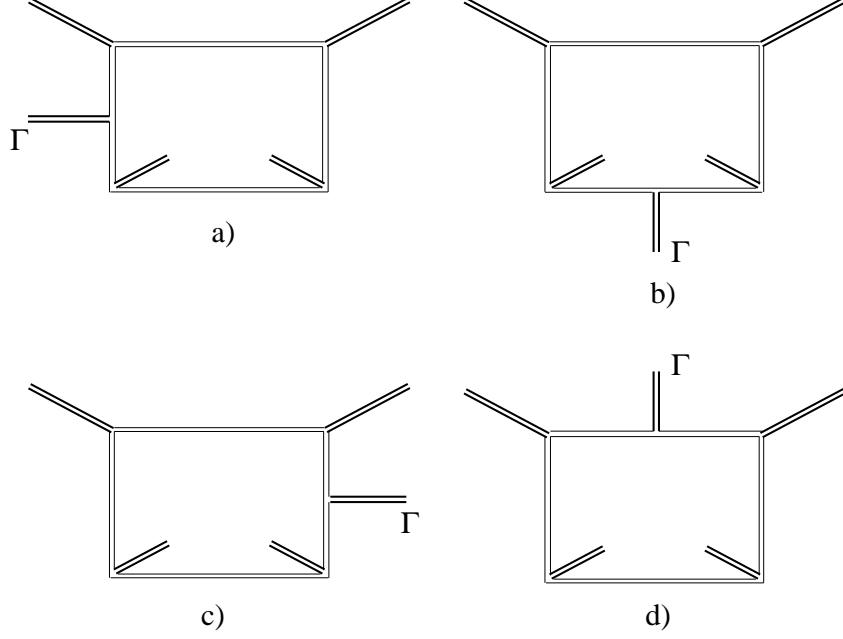


Figure 2: five-point function

At the level of the five-point function, the \mathcal{V} vertex can appear in between any of the four field strengths. The four diagrams corresponding to the $\delta\mathcal{V}_P$ variation are shown in Fig. 2a, ..., 2d. For each of them we have to consider $\delta_1\mathcal{V}_P$ and $\delta_2\mathcal{V}_P$. Let us start with $\delta_1\mathcal{V}_P$ as in Fig. 2a. It gives

$$\begin{aligned}
\delta_1\mathcal{V}_P^{(a)} \rightarrow & -\frac{i}{4} \int d^2\theta \, d^2\bar{\theta} \, \frac{d^4p_0 \, d^4p_1 \, d^4p_2 \, d^4p_3 \, d^4p_4}{(2\pi)^{20}} \, \delta(p_1 - p_0 + p_2 + p_3 + p_4 + p_0) \\
& \int d^4k \, \frac{e^{-\frac{i}{2}(p_1 \times p_2 - p_3 \times p_4)} e^{-ik \times (p_1 + p_2)}}{(k + p_1)^2 (k + p_0)^2 k^2 (k - p_4)^2 (k + p_1 + p_2)^2} (k + p_0)^2 e^{-\frac{i}{2}p_0 \times p_1} \\
& \text{Tr}[K(p_0)\mathbf{W}^\alpha(p_1 - p_0)\mathbf{W}_\alpha(p_2)] \, \text{Tr}[\bar{\mathbf{W}}^{\dot{\alpha}}(p_3)\bar{\mathbf{W}}_{\dot{\alpha}}(p_4)] \quad (27)
\end{aligned}$$

It is immediate to check that this expression exactly cancels the δ_1 variation in (22).

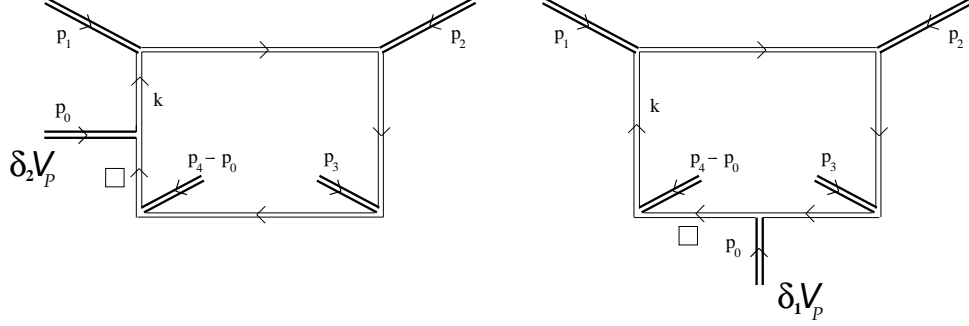


Figure 3: example of cancellation

Moreover it is rather simple to prove that the $\delta_2 \mathcal{V}_P$ term as in Fig. 2a compensates the terms from $\delta_1 \mathcal{V}_P$ as in Fig. 2b. Indeed with the labeling of momenta shown in Fig. 3 one obtains

$$\begin{aligned}
\delta_2 \mathcal{V}_P^{(a)} + \delta_1 \mathcal{V}_P^{(b)} &\rightarrow -\frac{i}{4} \int d^2\theta \, d^2\bar{\theta} \, \frac{d^4 p_0 \, d^4 p_1 \, d^4 p_2 \, d^4 p_3 \, d^4 p_4}{(2\pi)^{20}} \delta\left(\sum_{i=1}^4 p_i\right) \\
&\int \frac{d^4 k}{(k+p_1)^2 k^2 (k-p_4)^2 (k+p_1+p_2)^2} \left[-(k-p_0)^2 \frac{e^{-\frac{i}{2}k \times p_0} e^{\frac{i}{2}(k-p_4) \times (p_4-p_0)}}{(k-p_0)^2} \right. \\
&\quad \left. + (k-p_4+p_0)^2 \frac{e^{-\frac{i}{2}(k-p_4) \times p_0} e^{-\frac{i}{2}(p_4-p_0) \times k}}{(k-p_4+p_0)^2} \right] \\
&\text{Tr}[K(p_0) \mathbf{W}^\alpha(p_1) \mathbf{W}_\alpha(p_2)] \, \text{Tr}[\bar{\mathbf{W}}^{\dot{\alpha}}(p_3) \bar{\mathbf{W}}_{\dot{\alpha}}(p_4-p_0)] \\
&= 0
\end{aligned} \tag{28}$$

Continuing the calculation, one finds

$$\begin{aligned}
\delta_2 \mathcal{V}_P^{(b)} + \delta_1 \mathcal{V}_P^{(c)} &\rightarrow 0 \\
\delta_2 \mathcal{V}_P^{(c)} + \delta_1 \mathcal{V}_P^{(d)} &\rightarrow \delta_1 \Gamma_{PT}^{(2)} + \delta_2 \Gamma_{PT}^{(2)} \\
\delta_2 \mathcal{V}_P^{(d)} &\rightarrow \delta_2 \Gamma_{PT}^{(1)}
\end{aligned} \tag{29}$$

In this way we have shown that the terms produced from the variation of $\mathbf{W}^\alpha(p_1)$ and $\mathbf{W}_\alpha(p_2)$ in (22) are completely cancelled by the variation of the untwisted terms from the \mathcal{V} vertex in the five-point function. Following exactly the same pattern the twisted \mathcal{V} vertex produces a variation that compensates the variation from $\bar{\mathbf{W}}^{\dot{\alpha}}(p_3)$ and $\bar{\mathbf{W}}_{\dot{\alpha}}(p_4)$ in the four-point contribution in (22).

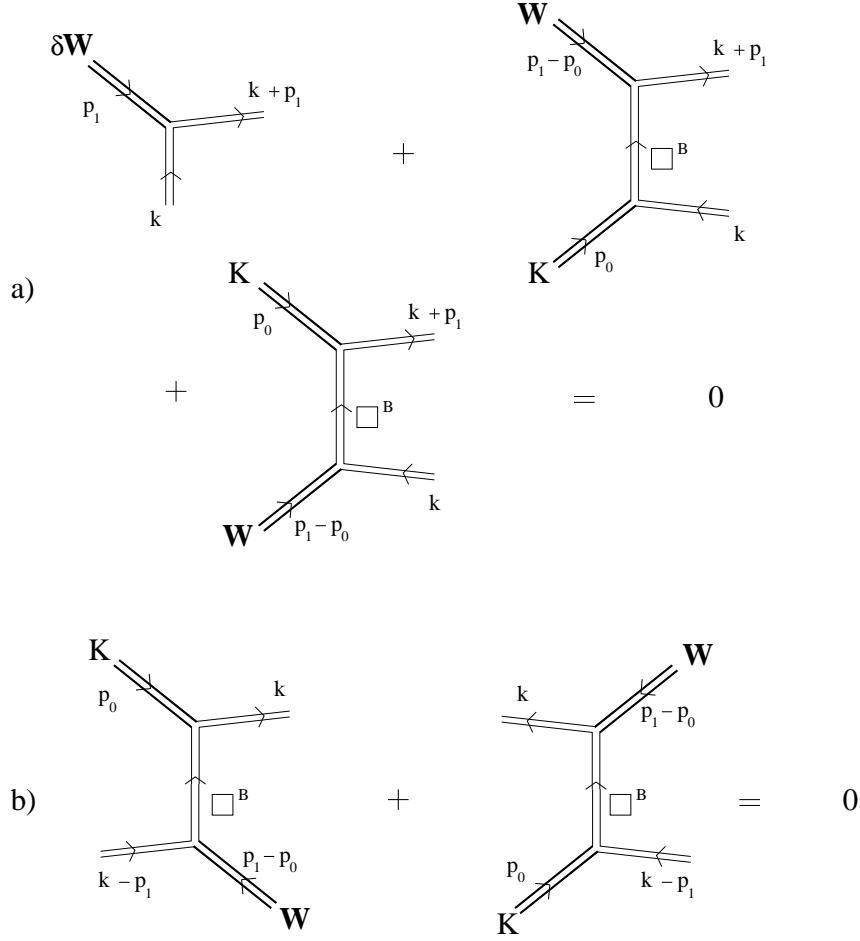


Figure 4: general cancellation mechanism

We can implement the cancellation graphically as in Fig. 4, which shows the interplay between twisted and untwisted vertices. In fact the previous argument can be generalized and the gauge invariance of the complete one-loop effective action can be obtained. The reasoning is the following: one rewrites the quadratic quantum Lagrangian in (7) in a manifestly background gauge invariant way

$$-\frac{1}{2g^2}\text{Tr}\left(V * \square^B * V - iV * \{\mathbf{W}^\alpha, \nabla_\alpha V\}_* - iV * \{\bar{\mathbf{W}}^\alpha, \bar{\nabla}_\alpha V\}_*\right) \quad (30)$$

with the definition $\square^B \equiv \frac{1}{2}\nabla_a^B \nabla^{Ba}$. As explained in [14] one can perform one-loop calculations using background covariant D -algebra and background covariant propagators. In particular, making a background gauge transformation on the background fields only, the first term of the action in (30) gives the variation in (25) with \square replaced by \square^B , as shown in Fig. 4b. The terms containing \mathbf{W} and $\bar{\mathbf{W}}$ are represented graphically in Fig. 4a, with covariant propagators $1/\square^B$. In this way, using the previous argument one establishes that the one-loop effective action is indeed background gauge invariant.

We conclude with the following remarks. We have checked that the noncommutative $\mathcal{N} = 4$ supersymmetric Yang-Mills theory can be treated perturbatively in a consistent and technically viable framework. Thus one can continue in the program and start computing correlation functions in perturbation theory. In [19, 20] gauge invariant operators have been constructed in the noncommutative theory through the introduction of appropriate Wilson lines

$$\mathcal{W}(x, C) = P_* \exp \left(i \int_0^1 d\sigma \frac{d\zeta^\mu}{d\sigma} A_\mu(x + \zeta(\sigma)) \right) \quad (31)$$

where P_* denotes generalized path ordering in terms of the $*$ -product and C is a straight line. The calculations that have been presented so far are in a component approach. We can extend them to supersymmetric theories in a full superspace formulation. Indeed the Wilson line can be represented in superfield language as

$$\mathcal{W}(x, C) = P_* \exp \left(i \int_0^1 d\sigma \frac{d\zeta^{\alpha\dot{\alpha}}}{d\sigma} \Gamma_{\alpha\dot{\alpha}}(x + \zeta(\sigma)) \right) \quad (32)$$

It can be used efficiently in perturbative calculations of correlation functions of gauge invariant operators.

The Wilson lines introduced in [19, 20] have been used to obtain a gauge-completion of the one-loop four-point function [21]. In the background field method approach that we have applied to derive the one-loop effective action, Wilson lines naturally appear as follows:

$$\nabla_a^B = \mathcal{W} * \partial_a \mathcal{W}^{-1} \quad \frac{1}{\square_B} = \mathcal{W} * \frac{1}{\square} * \mathcal{W}^{-1} \quad (33)$$

where \mathcal{W} is defined in (32). In this way contributions to the one-loop effective action can be rewritten as

$$\mathbf{W}^\alpha * \frac{1}{\square_B} * \nabla_B^\beta \bar{\nabla}_B^{\dot{\beta}} \bar{\mathbf{W}}^{\dot{\alpha}} * \frac{1}{\square_B} \dots \rightarrow \mathbf{W}^\alpha * \mathcal{W} * \frac{1}{\square} * \mathcal{W}^{-1} * \nabla_B^\beta \bar{\nabla}_B^{\dot{\beta}} \bar{\mathbf{W}}^{\dot{\alpha}} * \mathcal{W} * \frac{1}{\square} \dots \quad (34)$$

Acknowledgements

We acknowledge stimulating discussions with Ruggero Ferrari and Soo-Jong Rey. This work has been partially supported by INFN, MURST and the European Commission RTN program HPRN-CT-2000-00113 in which M.P., A.S. and D.Z. are associated to the University of Torino.

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